Optimal Control Systems

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1 Multivariable calcus

1.1 First order necessary condition for minimum

$$(\nabla f)^\top \cdot d \geq 0 \tag{1.1}$$

1.2 Second order necessary condition for minimum

$$(\nabla f)^\top \cdot d = 0 \tag{1.2}$$
$$d^\top \cdot \nabla^2 f \cdot d \geq 0 \tag{1.3}$$

Theorem 1.1. *Schwarz's theorem [1]*
If $\mathcal{C}^2 \ni f : (\mathbb{R}^2 \supset) E \to \mathbb{R}$ *then* $f_{12}(x,y) = f_{21}(x,y)$ *where* $(x,y) \in E^o$

2 Calculus of variations [2]

2.1 First order necessary condition for fixed end times and end states

$$\text{(Euler-Lagrange equation)} \qquad V_x - \frac{\mathrm{d}}{\mathrm{d}t} V_{x'} = V_x - p' = 0 \tag{2.1}$$

2.2 First order necessary condtion for variable end state and variable end time

$$V_x - p' = 0 \tag{2.2}$$
$$\left[p\delta x_f + (V - x'p)\delta t_f\right]\Big|_{t_f} = 0 \tag{2.3}$$

2.3 Special cases

2.3.1 V independent of x

$$\text{(Momenta is constant)} \qquad p' = 0 \tag{2.4}$$

2.3.2 V independent of t

$$\text{(Hamiltonian is constant)} \qquad \frac{\mathrm{d}}{\mathrm{d}t}H(t,x,x',p) = \frac{\mathrm{d}}{\mathrm{d}t}(px' - V) = 0 \tag{2.5}$$

2.4 Hamiltonian equations

$$x' = H_p \tag{2.6}$$
$$p' = -H_x \tag{2.7}$$

2.5 Second order condition

$$\text{(Legendre condition)} \qquad V_{x'x'} > 0 \tag{2.8}$$
$$V_{xx}V_{x'x'} - 2V_{xx'} > 0 \tag{2.9}$$

For fixed end point (2.9) becomes

$$\int_a^b \left(V_{xx} - \frac{\mathrm{d}}{\mathrm{d}t}V_{xx'} \right) (\delta x)^2 \,\mathrm{d}t + \int_a^b V_{x'x'}(\delta x')^2 \,\mathrm{d}t > 0 \tag{2.10}$$
$$\Rightarrow Q(x) := V_{xx} - \frac{\mathrm{d}}{\mathrm{d}t}V_{xx'} > 0 \tag{2.11}$$
$$\Rightarrow P(x) := V_{x'x'} > 0 \tag{2.12}$$

Ricatti equation

$$P(Q + w') = w^2 \tag{2.13}$$

Jacobi equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(Pv') = Qv, \quad w = -\frac{Pv'}{v} \tag{2.14}$$

Equation (2.11) is unnecessary if Ricatti or Jacobi equation has solution in the domain of interest and the problem at hand is a fixed boundary conditions problem.

2.6 Weak continuity

Defintion 2.1. *Weak convergence*
Let $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in vector space* $\mathcal{X}$*, and* $\mathcal{X}'$ *be set of all functionals on* $\mathcal{X}$*. If* $\forall f \in \mathcal{X}'$, $\lim_{x_n \to y} = f(y)$*, then* $x_n \xrightarrow[convergent]{weakly} y$

3 Lagrange multiplier theorem (for integral constraint)

$$\min \int_a^b V(t,x,x')dt \quad s.t. \quad \int_a^b W(t,x,x')dt = K_0 \tag{3.1}$$
$$\tag{3.2}$$

Let J and K be functional over $\mathcal{X}$ and ∂J, ∂K be weakly continuous. Then x^* is an extrema of K under the constraint $K = K_0$ implies either of (3.3) or (3.4) must hold.

$$\partial K(x_*, \partial x) = 0 \tag{3.3}$$
$$\partial J(x_*, \partial x) = \lambda\, \partial K(x_*, \partial x) \tag{3.4}$$

3.1 Necessary condition for minimizer

For all η s.t.

$$\partial K(x^*, \eta) = 0 \tag{3.5}$$

we must have

$$\partial J(x^*, \eta) \geq 0 \tag{3.6}$$

Note that above set of condition is similar to saying that along all feasible variation, first variation of J must be non-negative.

3.2 Integral constraint

$$J(x) = \int_a^b V(t, x, x')\mathrm{d}t \tag{3.7}$$

$$\min_x J(x) \quad s.t. \quad \int_a^b W(t, x, x') = 0 \tag{3.8}$$

Lagrange multiplier theorem corresponds to solving Euler Lagrange equation for $V(t, x, x') + \lambda K(t, x, x')$ for integral constraint. Refer [3] for converting integral constraint to non-integral constraint using a dummy variable.

3.3 Non-integral constraint

$$J(x) = \int_a^b V(t, x, x')\mathrm{d}t \tag{3.9}$$

$$\min_x J(x) \quad s.t. \quad W(t, x, x') = 0 \tag{3.10}$$

It corresponds to solving E-L equation for $V + \lambda(t)W$ along with constraint equation.

3.4 Integral E-L equation

$$V_{x'} - \int_a^b V_x = constant \tag{3.11}$$

4 Weirstrass-Erdman corner point condition for strong extrema

For each continuous region, E-L equation must hold and also momenta and Hamiltonian must be continuous at corner points (other points they are trivially continuous).

$$V_{x'}(t, x(t), x'(t))\Big|_{t_1^-}^{t_1^+} \delta x_1 + [V(t, x(t), x'(t) - x'(t)V_{x'}(t, x(t), x'(t))]\Big|_{t_1^-}^{t_1^+} \delta t_1 = 0 \tag{4.1}$$

5 Weirstrass excess function

Necessary condition for strong minima at all non-corner points

$$E(t,x,y,z) = V(t,x,z) - V(t,x,y) - (z-y)V_y(t,x,y) \geq 0 \tag{5.1}$$

6 Pontryagin's minimum principle [3]

$$\mathcal{H} = g(x,u) + p(t)^\top a(x,u,t) \tag{6.1}$$

$$J(u) = h(x_{t_f}, t_f) + \int_{t_0}^{t_f} g(x,u,t)dt \tag{6.2}$$

$$\dot{x} = \mathcal{H}_p \tag{6.3}$$

$$\dot{p} = -\mathcal{H}_x \tag{6.4}$$

$$H(t,x^*,p^*,u^*) \leq H(t,x^*,p^*,u) \tag{6.5}$$

$$\left[\frac{\partial h}{\partial x} - p\right]^\top\Bigg|_{t_f} \delta x_f + \left[\mathcal{H} + \frac{\partial h}{\partial t}\right]\Bigg|_{t_f} \delta t_f = 0 \tag{6.6}$$

Additional necessary conditions are as follows:

- If final time is free and Hamiltonian does not explicitly depends on time, then

$$H(x^*,u^*,p^*) = 0 \tag{6.7}$$

$$\tag{6.8}$$

- If final time is fixed and Hamiltonian does not explicitly depend on time, then

$$H(x^*,u^*,p^*) = constant \tag{6.9}$$

7 Hamilton-Jacobi-Bellman equation

$$0 = J_t^*(x,t) + \inf_{u\in\mathcal{U}}[g(x,u,t) + J_x(x,t)a(x,u,t)] = \inf_u[g(x,u,t) + \frac{d}{dt}J(t,x)] \tag{7.1}$$

$$J(x(t_f),t_f) = h(x(t_f),t_f) \tag{7.2}$$

8 Linear quadratic regulator

Finite horizon

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{8.1}$$

$$J(x,t) = \frac{1}{2}x(t_f)^\top Hx(t_f) + \frac{1}{2}\int_{t_0}^{t_f}(x(t)^\top Q(t)x(t) + u(t)^\top R(t)u(t))dt \tag{8.2}$$

$$p(t) = K(t)x(t);\ K(t) \in S^{nxn} \tag{8.3}$$

$$u(t) = -R^{-1}B^\top(t)K(t)x(t) \tag{8.4}$$

$$-\dot{K} = KA + A^\top K + Q - KBR^{-1}B^\top K \tag{8.5}$$

$$\text{Hamiltonian matrix} \triangleq \mathcal{H} = \begin{bmatrix} A & -BR^{-1}B^\top \\ -Q & -A^\top \end{bmatrix} \tag{8.6}$$

Infinite horizon

$$H = 0, \text{all matrices are time invariant} \tag{8.7}$$
$$0 = KA + A^\top K + Q - KBR^{-1}B^\top K \tag{8.8}$$

Refer Appendix D of [4] for techniques to find K of algebraic Ricatti equation using Hamiltonian matrix.

8.1 Linear tracking problem

$$\begin{aligned} J(x,t) = &\frac{1}{2}(x(t_f) - r(t_f))^\top H(x(t_f) - r(t_f)) \\ &+ \frac{1}{2}\int_{t_0}^{t_f}((x(t) - r(t))^\top Q(t)(x(t) - r(t)) + u(t)^\top R(t)u(t))dt \end{aligned} \tag{8.9}$$

$$p(t) = K(t)x(t) + s(t) \tag{8.10}$$
$$\dot{s} = -A^\top s + KBR^{-1}B^\top s + Qr \tag{8.11}$$
$$s(t_f) = -Hr(t_f) \tag{8.12}$$

8.2 Minimum time problem

Refer to Section 5.4, page 240 of [3] and Section 5.6 for a discussion on singular intervals.

9 Reference

[1] W. Rudin. *Principles of mathematical analysis*. 3d ed. International series in pure and applied mathematics. McGraw-Hill, 1976.

[2] D. Liberzon. *Calculus of Variations and Optimal Control Theory*. Princeton University Press, 2012.

[3] D. E. Kirk. *Optimal control theory: An introduction*. Dover Publications, 2004.

[4] Graham C. Goodwin, Stefan F. Graebe, and Mario E. Salgado. *Control System Design*. Prentice Hall, 2000. ISBN: 0139586539,9780139586538.