Differential Equations

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1 Linear ODE

$$a_0(x)y^{(n)}(x) + ... + a_n(x)y(x) = b(x) \tag{1}$$

2 Separable ODE

$$M(x) + N(y)\frac{dy}{dx} = 0 \tag{2}$$

3 Homogeneous ODE

3.1 Homogeneous function

$$f(tx_1, ..., tx_n) = t^d f(x_1, ..., x_n) \tag{3}$$

3.2 Conversion to separable ODE

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{4}$$

$$\frac{y}{x} = v \tag{5}$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \tag{6}$$

$$\Rightarrow x^d M(1,v) + x^d N(1,v)\left(v + x\frac{dv}{dx}\right) = 0 \tag{7}$$

$$\Rightarrow M(1,v) + N(1,v)\left(v + x\frac{dv}{dx}\right) = 0 \tag{8}$$

$M(x,y), N(x,y)$ are homogeneous functions of same order.

4 Exact ODE

$$M(x,y) + N(x,y)y^1 = 0 \tag{9}$$
$$\frac{\partial u(x,y)}{dx} = M(x,y) \tag{10}$$
$$\frac{\partial u(x,y)}{dy} = N(x,y) \tag{11}$$
$$\Rightarrow \frac{\partial u(x,y)}{dx}dx + \frac{\partial u(x,y)}{dy}dy = 0 \tag{12}$$
$$\Rightarrow du = 0 \tag{13}$$
$$\Rightarrow u(x,y) = c \tag{14}$$

5 Closed form

$$\frac{dM(x,y)}{dy} = \frac{dN(x,y)}{dx} \tag{15}$$

Exact ODE is always in closed form. If D is convex, then closed form is exact.

6 Integrating factors

If $M_y \neq N_x$ but $(\mu M)_y = (\mu N)_x$ where $\mu(x,y)$ is an integrating factor. Generally, we assume integrating factor is a function of one variable only.

$$\mu M_y + \mu_y M = \mu N_x + \mu_x N \tag{16}$$
$$\Rightarrow \mu M_y = \mu N_x + \mu_x N \tag{17}$$
$$\Rightarrow \frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu \tag{18}$$

If the equation (18) is linearly separable, we are done or else we look for

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu \tag{19}$$

and see if this is linearly separable. Then, using μ we get an exact ODE.

7 First order linear ODE

$$\frac{dy}{dx} + p(x)y = g(x) \tag{20}$$
$$\mu = exp\left(\int p(x)dx\right) \tag{21}$$

8 Bernoulli's ODE

$$y^{(1)} + p(x)y = g(x)y^n \tag{22}$$
$$u = \frac{1}{y^{n-1}} \tag{23}$$
$$\Rightarrow \frac{du}{dx} = \frac{1-n}{y^n}\frac{dy}{dx} \tag{24}$$
$$\Rightarrow \frac{1}{n-1}\frac{du}{dx} + p(x)u(x) = q(x) \tag{25}$$

9 Picard's iteration method

$$y^{(1)} = f(t, y),\ y(0) = 0 \tag{26}$$

Suppose $y = \phi(t)$ is a solution.

$$\phi(t) = \int_0^t f(s, \phi(s))ds \tag{27}$$

$$Let\ \phi_0(t) = 0 \tag{28}$$

$$\phi_1(t) = \int_0^t f(s, \phi_0(s))ds \tag{29}$$

If $\phi_{n+1}(t) = \phi_n(t)$, that is the solution of IVP.

10 Second order linear ODE

11 Cauchy-Euler DE

$$x^2y^{(2)} + axy^{(1)} + by = 0 \tag{30}$$

$$y = x^m;\ x > 0 \tag{31}$$

m is a solution of quadratic equation. If roots are equal, we use variation of parameter to obtain the second root. Second root turns out to be $ln(x) * x^m$. [Ref. Pg. 120 1-14]

12 Constant coefficient ODE

$$y^{(n)} + a_1y^{(n-1)} + ... + a_ny = r(x) \tag{32}$$

Find solution to

$$y^{(n)} + a_1y^{(n-1)} + ... + a_ny = 0 \tag{33}$$

by substituting $y^{(n)} = D^n$. For repeated roots, multiply the solution with x. To find a general solution, let

$$y = v_1y_1 + ... + v_ny_n \tag{34}$$

$$\begin{bmatrix} y_1 & y_2 & ... & y_n \\ y_1^{(1)} & y_2^{(1)} & ... & y_n^{(1)} \\ ... & ... & ... & ... \\ y_1^{(n-1)} & y_2^{(n-1)} & ... & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \\ ... \\ v_n^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ ... \\ r(x) \end{bmatrix} \tag{35}$$

$$W(x) = det\left(\begin{bmatrix} y_1 & y_2 & ... & y_n \\ y_1^{(1)} & y_2^{(1)} & ... & y_n^{(1)} \\ ... & ... & ... & ... \\ y_1^{(n-1)} & y_2^{(n-1)} & ... & y_n^{(n-1)} \end{bmatrix}\right) = W(0)exp\left(-\int_0^x a_1dt\right) \tag{36}$$

$$\tag{37}$$

Use equation (36) to obtain denominator while solving the equation (35) using Cramer's rule.

13 Laplace transform

14 Boundary value problem

Values of solution are specified at multiple operating points known as the boundary points. It may or may not be possible to come up with a non-trivial solution if the BVP is not properly specified.

15 Partial differential equation

15.1 Variable separable method

$$u(x,t) = X(x)T(t) \tag{38}$$

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