Complex variables

Anurag
anuragg.in@gmail.com
www.anuragg.in

1 Analytic functions

If the function is continuously differentiable.

1.1 Cauchy-Reimann equations

Let $\mathbb{G}$ be a region in $\mathbb{C}$ and let u and v be a function on $\mathbb{G}$ with continuous partial derivative. Then, $f(z) = u(z) + iv(z)$ on $\mathbb{G}$ is analytic if and only if the Cauchy-Reimann equations hold, i.e.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2}$$

$$\tag{3}$$

1.2 Properties

Theorem 1. *If a function $f(z)$ is analytic at a point, then its derivative of all orders are analytic at the same point.*

Theorem 2. *If f is entire and bounded in $\mathbb{C}$, then $f(z)$ is constant throughout the plane.*

2 Cauchy's integral theorem

Let $f(z)$ be analytic on and inside a closed contour C and let $f'(z)$ be also continuous on and inside C, then

$$\oint_C f(z)dz = 0 \tag{4}$$

Definition 1. *Positively oriented closed curve A planar simple closed curve such that when traveling on it, one always has interior of the curve on its left.*

Corollary 1. *Let $C, C_1, C_2, ..., C_n$ be positively oriented closed contours, where C_i $\forall i$ lies inside C and each C_i lies outside of each other. Let f(z) be analytic on the set $C \cup int(C) \setminus int(C_1) \setminus ... \setminus int(C_n)$, then*

$$\oint_C f(z)dz = \sum_k \oint_{C_k} f(z)dz \tag{5}$$

3 Cauchy's integral formula

Let $f(z)$ be analytic on and inside a positively oriented closed contour C and let z be any point inside C, then

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta \tag{6}$$

Corollary 2. *Let z be a point in the set $C \cup int(C) \setminus int(C_1) \setminus ... \setminus int(C_n)$ where C, C_i and $f(z)$ have the same condition as in Corollary 1, then*

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \sum_k \oint_{C_k} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{7}$$

Corollary 3. *Assuming differentiation under the integration sign is legitimately defined,*

$$f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \tag{8}$$

[Ref. example on Page 56 of additional material slides of integration.pdf]

Corollary 4. *Assuming differentiation under the integration sign is legitimately defined,*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \tag{9}$$

4 Taylor's theorem

Let $f(z)$ be analytic at all points within a circle C_0 with center z_0 and radius ρ_0. Then, for every point z within C_0, we have

$$f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) \tag{10}$$

where the power series converges inside the disc.

5 Laurent series

Let $f(z)$ be analytic in the annular region $ann(a; R_1, R_2)$. Then

$$f(z) = \sum_{i=-\infty}^{\infty} a_n (z - a)^n \tag{11}$$

$$a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - a)^{n+1}} \tag{12}$$

Convergence over annular region. γ is any closed curve in annular region.

[Refer example on NPTEL slides]

6 Residue theorem

Definition 2 (Residue)**.** *The coefficient a_{-1} is the residue of $f(z)$ at $z = a$ or $Res(f; a)$.*

Theorem 3 (Residue theorem)**.** *Let $f(z)$ be analytic on a region $\mathbb{G}$ except for singularities at $a_1, ..., a_n$. If γ is a closed piecewise smooth curve in $\mathbb{G}$ which does not pass through any of the singular points, then*

$$\frac{1}{2\pi i} \oint_\gamma f(z) dz = \sum_k Res(f; a_k) \tag{13}$$

7 Reference

http://nptel.ac.in/courses/111107056/21